

# Quantifying residual finiteness of arithmetic groups

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## Abstract

The normal Farb growth of a group quantifies how well-approximated the group is by its finite quotients. We show that any  $S$ -arithmetic subgroup of a higher rank Chevalley group  $G$  has normal Farb growth  $n^{\dim(G)}$ .

keywords: *Arithmetic groups, normal Farb growth, residual finiteness*

## 1 Introduction

The quantification of residual finiteness, begun in [B10], seeks to describe how well a residually finite group is approximated by its finite quotients. This is measured by the normal Farb growth of the group. During a geometry seminar at Yale University in December 2009, Daniel Mostow asked the following question:

**Question 1.1.** (D. Mostow) Does asymptotic information of residual finiteness characterize arithmetic subgroups of a given linear algebraic group?

This paper presents a first major step towards answering this question, by showing that in a fixed Chevalley group  $G$ , all  $S$ -arithmetic subgroups share the same normal Farb growth, and moreover this growth is  $n^{\dim(G)}$ . Note that for us, a Chevalley group will be a split simple algebraic group that is not necessarily simply-connected.

To state our results more precisely, we need some notation. Let  $\Gamma$  be a finitely generated, residually finite group, and let  $X$  be a finite generating set for  $\Gamma$ . For  $\gamma \in \Gamma$ , let  $\|\gamma\|_X$  denote the word length of  $\gamma$  with respect to  $X$ . Define

$$D_\Gamma(\gamma) := \min\{|Q| : Q \text{ is a finite quotient of } \Gamma \text{ where } \gamma \neq 1\},$$

and

$$F_{\Gamma,X}(n) := \max\{D_\Gamma(\gamma) : \|\gamma\|_X \leq n\}.$$

The function  $F_{\Gamma, X}$  is called the *normal Farb growth function*. It is known that the asymptotic behavior of  $F_{\Gamma, X}$  is independent of  $X$  (see Section 2). The asymptotic growth of this function is called the *normal Farb growth* of  $\Gamma$ .

The main results of this paper characterize the normal Farb growth of  $S$ -arithmetic groups in Chevalley groups. We use the term  $S$ -arithmetic subgroup of  $G$  to denote any subgroup of  $G(\mathbb{C})$  which is commensurable with  $G(\mathcal{O}_{K, f})$ , where  $K \subset \mathbb{C}$  is a number field,  $\mathcal{O}_K$  is its ring of integers, and  $f \in \mathcal{O}_K \setminus \{0\}$ . That is, it is an  $S$ -arithmetic subgroup of  $G$  in the usual definition for some number field  $K$  and some finite set  $S$  of places of  $K$  which contains the archimedean ones, but we allow  $K$  and  $S$  to vary. The ingredients used include the structure theory of split semi-simple group schemes, results on the congruence subgroup problem, Moy-Prasad filtrations, Selberg's Lemma, the prime number theorem, and the Cebotarev density theorem. Furthermore, we use in an essential way the results of Lubotzky-Mozes-Ragunathan [LMR01].

**Theorem 1.2.** *Let  $G$  be a Chevalley group of rank at least 2,  $K$  be a number field, and  $f \in \mathcal{O}_K \setminus \{0\}$ . If  $\Gamma$  is a finitely generated subgroup of  $G(\mathbb{C})$  with the property that  $\Gamma \cap G(\mathcal{O}_{K, f})$  is of finite-index in  $G(\mathcal{O}_{K, f})$ , then its normal Farb growth is bounded below by  $n^{\dim(G)}$ .*

It is interesting to ask whether an analogous result holds in rank 1. So far, the normal Farb growth of a nonabelian free group has been bounded below by  $n^{2/3}$  (see [KM10]).

**Theorem 1.3.** *Let  $G$  be a Chevalley group,  $K$  be a number field, and  $f \in \mathcal{O}_K \setminus \{0\}$ . If  $\Gamma$  is a finitely generated subgroup of  $G(\mathbb{C})$  with the property that  $\Gamma \cap G(\mathcal{O}_{K, f})$  is of finite-index in  $\Gamma$ , then its normal Farb growth is bounded above by  $n^{\dim(G)}$ .*

As a corollary of Theorems 1.2 and 1.3 we have the following result.

**Corollary 1.4.** *Let  $G$  be a Chevalley group of rank at least 2. Then the normal Farb growth of every  $S$ -arithmetic subgroup of  $G$  is precisely  $n^{\dim(G)}$ .*

This result is surprising since in general, if  $\Delta$  has finite-index in  $\Gamma$ , we cannot hope for  $F_\Gamma \approx F_\Delta$  (see Example 2.5 at the end of Section 2). Instead, the most general result in this direction is  $F_\Gamma(n) \preceq (F_\Delta(n))^{\lfloor \Gamma:\Delta \rfloor}$  (see [B10, Lemma 1.3]).

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## 2 Preliminaries

Let  $\Gamma$  be a finitely generated, residually finite group. For  $\gamma \in \Gamma \setminus \{1\}$  we define  $\mathcal{Q}(\gamma, \Gamma)$  to be the set of finite quotients of  $\Gamma$  in which the image of  $\gamma$  is non-trivial. We say that these quotients detect  $\gamma$ . Since  $\Gamma$  is residually finite, this set is non-empty, and thus the natural number

$$D_\Gamma(\gamma) := \min\{|\mathcal{Q}| : \mathcal{Q} \in \mathcal{Q}(\gamma, \Gamma)\}$$

is defined and positive for each  $\gamma \in \Gamma \setminus \{1\}$ . For a fixed finite generating set  $X \subset \Gamma$  we define

$$F_{\Gamma, X}(n) := \max\{D_\Gamma(\gamma) : \gamma \in \Gamma, \|\gamma\|_X \leq n, \gamma \neq 1\}.$$

For two functions  $f, g : \mathbb{N} \rightarrow \mathbb{N}$  we write  $f \preceq g$  if there exists a natural number  $M$  such that  $f(n) \leq Mg(Mn)$ , and we write  $f \approx g$  if  $f \preceq g$  and  $g \preceq f$ . We will also write  $f \succeq g$  for  $g \preceq f$  and in the case when  $f \approx g$  does not hold we write  $f \not\approx g$ .

It was shown in [B10] that if  $X, Y$  are two finite generating sets for the residually finite group  $\Gamma$ , then  $F_{\Gamma, X} \approx F_{\Gamma, Y}$ . Since we will only be interested in asymptotic behavior, we let  $F_\Gamma$  be the equivalence class (with respect to  $\approx$ ) of the functions  $F_{\Gamma, X}$  for all possible finite generating sets  $X$  of  $\Gamma$ . Sometimes, by abuse of notation,  $F_\Gamma$  will stand for some particular representative of this equivalence class, constructed with respect to a convenient generating set.

We will need to use the following auxiliary function in our proofs. For any natural number  $k$ , we define

$$D_\Gamma^k(\gamma) := D_\Gamma(\gamma^k) \text{ and } F_{\Gamma, X}^k(n) := \max\{D_\Gamma^k(\gamma) : \gamma \in \Gamma, \|\gamma\|_X \leq n, \gamma^k \neq 1\}.$$

The next lemma, which is a consequence of Selberg's Lemma (see [A87]), reveals the potential utility of  $F_{\Gamma, X}^k$ .

**Lemma 2.1.** *Let  $\Gamma$  be an infinite linear group generated by a finite set  $X$  and let  $k$  a natural number. Then  $F_{\Gamma, X} \approx F_{\Gamma, X}^k$ .*

*Proof.* The inequality  $F_{\Gamma, X}^k(n) \leq F_{\Gamma, X}(kn)$  is straightforward. It suffices to prove  $F_{\Gamma, X}(n) \leq F_{\Gamma, X}^k(n)$  for all but finitely many  $n$ . Let  $\gamma_n$  be an element such that  $D_\Gamma(\gamma_n) = F_{\Gamma, X}(n)$  and  $\|\gamma_n\|_X \leq n$ . If  $\gamma_n^k \neq 1$ , then  $D_\Gamma(\gamma_n) \leq D_\Gamma^k(\gamma_n)$ , giving  $F_{\Gamma, X}(n) \leq F_{\Gamma, X}^k(n)$ . The proof will be complete if we show that  $\gamma_n^k = 1$  holds for only finitely many  $n$ . Suppose otherwise, then by Selberg's Lemma, there exists a finite-index normal subgroup  $\Delta$  of  $\Gamma$  that is torsion-free, and in particular  $\gamma_n \notin \Delta$  for infinitely many  $n$ . Since  $F_{\Gamma, X}(n)$  is non-decreasing in  $n$ , it must be bounded by  $[\Gamma : \Delta]$ , but this contradicts the infinitude of  $\Gamma$ .  $\square$

**Corollary 2.2.** *If  $\Gamma$  is an infinite linear group and  $X, Y$  are finite generating sets for  $\Gamma$ , then  $F_{\Gamma, X}^k \approx F_{\Gamma, Y}^k$ .*

As with the function  $F$ , we will denote the asymptotic equivalence class of  $F_{\Gamma,X}^k$  as  $X$  varies by  $F_{\Gamma}^k$ . The following example shows that the linearity assumption cannot be dropped from Lemma 2.1.

**Example 2.3.** Let  $\Gamma$  be the Lamplighter group  $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$ . Set  $\Delta = \bigoplus_{i \in \mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$  to be the base group of  $\Gamma$  so  $\Gamma/\Delta \cong \mathbb{Z}$ . It is easy to see that for any generating set  $X$  of  $\Gamma$ , we have  $F_{\Gamma,X}^2(n) \approx F_{\mathbb{Z}}(n)$ . Thus  $F_{\Gamma,X}^2(n) \approx \log(n)$  by [B10, Corollary 2.3]. We now prove that  $F_{\Gamma}(n) \succeq (\log(n))^2$ , so in particular  $F_{\Gamma} \not\approx F_{\Gamma,X}^2$ .

*Proof.* Let  $\delta_i \in \Delta$  be the element given by the  $i$ th Kronecker delta function. For  $k$  a natural number greater than 4, set  $\gamma_k := \delta_1 + \delta_{\text{lcm}(1,\dots,k)}$ . Let  $\phi : \Gamma \rightarrow P$  be a homomorphism to a finite quotient of  $\Gamma$  that realizes  $D_{\Gamma}(\gamma_k)$ . We first claim that if  $\delta_1 + \delta_{1+n} \in \ker \phi$  for  $n \in \mathbb{N}$ , then  $n \geq k$ . Indeed, a simple calculation shows that  $\delta_1 + \delta_{1+mn} \in \ker \phi$  for any  $m \in \mathbb{N}$ . If  $n \leq k$ , we have that  $\text{lcm}(1,\dots,k)$  is a multiple of  $n$ , so  $\delta_1 + \delta_{\text{lcm}(1,\dots,k)} \in \ker \phi$ , which is impossible.

Next, we claim that the set  $S := \{(\delta_n, t) : n, t \in \{1, \dots, \lfloor k/4 \rfloor\}\} \subseteq \Gamma$  injects into  $P$  through  $\phi$ . Suppose not, then  $(\delta_n, t)(\delta_{n'}, t')^{-1} \in \ker \phi$  for  $t, t', n, n' \in \{1, \dots, \lfloor k/4 \rfloor\}$  with  $(\delta_n, t) \neq (\delta_{n'}, t')$ . Set  $\alpha = (\delta_n, t)(\delta_{n'}, t')^{-1} = (\delta_n + \delta_{n'+t-t'}, t-t')$ . If  $t-t' = 0$ , then by our first claim  $n = n'$  or  $|n| - |n'| \geq k$ . If  $n = n'$ , then  $\alpha = (0, 0)$ , while the latter possibility contradicts  $|n| - |n'| \leq k/2$ . If  $t-t' \neq 0$ , because  $\alpha \delta_i \alpha^{-1} \delta_i^{-1} \in \ker \phi$  for all  $i$ , we have  $\delta_{1+t-t'} + \delta_1 \in \ker \phi$ , where by our first claim,  $|t-t'| \geq k$ , however  $|t-t'| \leq k/2$ . Our second claim is now shown.

Since  $S$  injects into  $P$ , we have  $|P| \geq \lfloor k/4 \rfloor^2$ . Fix a finite generating set  $X$  for  $\Gamma$ , by the prime number theorem, there exists a natural number  $M$  such that  $\|\gamma_k\|_X \leq M3^k$ . Set  $k = \lfloor \log_3(n) \rfloor$ , then because  $F_{\Gamma}$  is increasing we have, for sufficiently large  $n$ ,

$$F_{\Gamma}(Mn) \geq F_{\Gamma}(M3^k) \geq F_{\Gamma}(\|\gamma_k\|_X) \geq \lfloor k/4 \rfloor^2 \geq \frac{1}{32} \left[ \frac{\log(n)}{\log(3)} \right]^2.$$

□

**Lemma 2.4.** Let  $\Gamma, \Delta$  be finitely generated and residually finite. Then

- If  $\Delta \subset \Gamma$ , then  $F_{\Delta} \preceq F_{\Gamma}$ .
- If  $f : \Delta \rightarrow \Gamma$  is surjective with finite kernel, then  $F_{\Delta} \preceq F_{\Gamma}$ . If moreover  $\ker(f)$  is central in  $\Delta$  and  $\Gamma$  is linear, then  $F_{\Delta} \approx F_{\Gamma}$ .

*Proof.* The first assertion is [B10, Lemma 1.1]. Consider the second assertion. The inequality  $F_{\Delta} \preceq F_{\Gamma}$  is straightforward. Assuming now that  $\ker(f)$  is central in  $\Delta$ , we will show  $F_{\Delta}^k \succeq F_{\Gamma}$ , where  $k = |\ker(f)|$ . To that end, fix a finite generating set  $X$  for  $\Delta$  and use its image for  $\Gamma$ . Construct  $F_{\Delta}$  and  $F_{\Gamma}$  with respect to these generating sets. Let  $g \in \Delta$ ,  $g^k \neq 1$ .

Since  $g^k = (zg)^k$  for all  $z \in \ker(f)$ , we see that  $\ker(f)N$  is a normal subgroup of  $\Delta$  not containing  $g$ . Thus  $D_\Delta^k(g) \geq D_\Gamma(f(g))$  for all  $g \in \Delta$  with  $g^k \neq 1$ . We now need to handle torsion elements in  $\Gamma$ .

For each natural number  $n$ , let  $\gamma_n \in \Gamma$  be an element satisfying  $D_\Gamma(\gamma_n) = F_\Gamma(n)$  and  $\|\gamma_n\| \leq n$ . Since  $f$  is surjective and by our choice of generating sets, there exists  $g_n \in \Delta$  such that  $f(g_n) = \gamma_n$  and  $\|g_n\| \leq n$ . Then if  $g_n^k = 1$  for infinitely many  $n$ , then  $\gamma_n^k = 1$  for infinitely many  $n$ . Following the Selberg Lemma application from Lemma 2.1, we see that  $\Gamma$  is finite, which is impossible. Thus,  $g_n^k \neq 1$  for all but finitely many  $n$ . For such  $n$ , we have  $D_\Delta^k(g_n) \geq D_\Gamma(f(g_n))$  and hence  $F_\Delta^k(n) \succeq F_\Gamma(n)$ .  $\square$

We finish the preliminaries section with an example that illustrates that normal Farb growth of a group may be different from that of a finite index subgroup.

**Example 2.5.** Let  $Q$  be the subgroup of  $\mathrm{GL}_2(\mathbb{Z})$  generated by

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let  $\Delta = \mathbb{Z} \times \mathbb{Z}$  and set  $\Gamma = \Delta \rtimes Q$ , where  $Q$  acts on  $\Delta$  via the standard action of  $\mathrm{GL}_2(\mathbb{Z})$ . Because  $Q$  is finite,  $\Gamma$  contains  $\Delta$  as a subgroup of finite-index. Further,  $F_\Delta(n) \approx \log(n)$  by [B10, Corollary 2.3]. We now prove that  $F_\Gamma(n) \succeq (\log(n))^2$ .

*Proof.* Let  $X$  be a generating set for  $\Gamma$  containing  $(1,0)$  and  $(0,1)$  in  $\Delta$ . Set  $\gamma_k$  to be  $(\mathrm{lcm}(1, \dots, k), 0) \in \Delta$ . By the prime number theorem, there exists a natural number  $M$  such that  $\|\gamma_k\|_X \leq M3^k$ . Let  $\phi : \Gamma \rightarrow P$  be a homomorphism to a finite quotient of  $\Gamma$  that realizes  $D_\Gamma(\gamma_k)$  and set  $V = \ker \phi \cap \Delta$ . We first construct a subgroup of  $V$  of the form  $d\mathbb{Z} \times d\mathbb{Z}$  for some natural number  $d$ . Consider the intersection of  $V$  with  $\mathbb{Z} \times 0$ . This is a subgroup of  $\mathbb{Z}$ , hence is isomorphic to  $d\mathbb{Z}$  for some natural number  $d$ . Thus we have  $d\mathbb{Z} \times 0 \subset V$ , and conjugating by  $B$  we also find  $0 \times d\mathbb{Z}$  is in  $V$ .

Next, we claim that the index of  $d\mathbb{Z} \times d\mathbb{Z}$  in  $V$  is at most 4: Let  $(a,b) \in V$ . Then  $(2a,0) = (a,b) + A(a,b)A^{-1} \in V$ , and similarly  $(2b,0) \in V$ , so  $2a, 2b \in d\mathbb{Z}$ , and hence  $2(a,b) \in d\mathbb{Z} \times d\mathbb{Z}$ , which shows that every element of  $V/d\mathbb{Z} \times d\mathbb{Z}$  has order (at most) 2. But  $V$  is a free abelian group of rank 2, so  $V/d\mathbb{Z} \times d\mathbb{Z}$  is generated by two elements, and the claim follows. We conclude that  $d^2 = [\Delta : d\mathbb{Z} \times d\mathbb{Z}] = [\Delta : V][V : d\mathbb{Z} \times d\mathbb{Z}] \leq 4[\Delta : V]$ , giving  $|P| \geq \frac{1}{4}d^2$ .

Finally, since  $\gamma_k \notin \ker(\phi)$ , we must have that  $d \geq k$ . Hence,  $F_\Gamma(M3^k) \geq D_\Gamma(\gamma_k) \geq \frac{1}{4}k^2$ . Set  $k = \lfloor (\log_3(n)) \rfloor$ , then because  $F_\Gamma$  is increasing we have, for sufficiently large  $n$ ,

$$F_\Gamma(Mn) \geq F_\Gamma(M3^k) \geq \frac{1}{4}k^2 \geq \frac{1}{16} \left( \frac{\log(n)}{\log(3)} \right)^2,$$

giving  $F_\Gamma(n) \succeq (\log(n))^2$ , as desired.  $\square$

### 3 Lower bounds

Let  $G$  be a Chevalley group, i.e. a split simple group scheme defined over  $\mathbb{Z}$ , and let  $\mathfrak{g}$  be its Lie-algebra. Note that we do not assume that  $G$  is simply-connected. For a natural number  $m$ , we put  $G(m) = G(\mathbb{Z}/m\mathbb{Z})$ . For a while, we will focus attention on the powers of a single prime  $p$ , and to lighten the notation we put  $G_k = G(\mathbb{Z}/p^k\mathbb{Z})$ .

Recall from [SGA3, exp.1, 2.3.3+2.3.6] the definition of the center  $Z(G)$  of  $G$ . It is the subfunctor of  $G$ , which assigns to each scheme  $S$  the following subgroup of  $G(S)$

$$Z(G)(S) := \{g \in G(S) \mid \forall S' \rightarrow S : \text{Ad}(g)|_{G(S')} = \text{id}_{G(S')}\}$$

where  $\text{Ad}(g)|_{G(S')}$  denotes the automorphism of  $G(S')$  provided by conjugation by the image of  $g$  under the natural map  $G(S) \rightarrow G(S')$ .

It is shown in [SGA3, exp.22, 4.1.8] that the functor  $Z(G)$  is representable by a closed  $\mathbb{Z}$ -subgroup-scheme of  $G$ , which is finite and diagonalizable. As such,  $Z(G)$  is a product of finitely many groups schemes, each isomorphic to  $\mu_n$  for some  $n$ , where  $\mu_n$  is the group scheme of  $n$ -th roots of unity. In particular,  $Z(G)$  is etale over  $\mathbb{Z}[\text{ord}(Z(G))^{-1}]$ . See [SGA3, exp.8, 2.1].

From the definition it is obvious that  $Z(G)(S) \subset Z(G(S))$ . We will show that there exists  $f \in \mathbb{Z} \setminus \{0\}$  such that if  $S$  lies over  $\text{Spec}(\mathbb{Z}_f)$ , then  $Z(G)(S) = Z(G(S))$ . The main ingredient in this proof is the following lemma, which asserts the existence of a strongly regular section of the split maximal torus in  $G$  over  $\text{Spec}(\mathbb{Z}_f)$ .

**Lemma 3.1.** *Let  $T \subset G$  be a split maximal torus. There exists  $f \in \mathbb{Z} \setminus \{0\}$  and a point  $s \in T(\mathbb{Z}_f)$  such that*

$$\text{Cent}(s, G \times \text{Spec}(\mathbb{Z}_f)) = T \times \text{Spec}(\mathbb{Z}_f).$$

*Proof.* Consider the closed subscheme of  $T$  given by

$$\bigcup_{\alpha \in R(T, G)} \ker(\alpha) \cup \bigcup_{w \in W} T^w$$

where  $R(T, G)$  is the set of roots of  $T$  in  $G$  and  $W = \text{Norm}(G, T)/T$  is the Weyl group. Let  $U$  be its complement in  $T$ . Then  $U \rightarrow T$  is an open immersion, which when composed with an isomorphism  $T \cong \mathbb{G}_m^r$  and the open immersion  $\mathbb{G}_m^r \rightarrow \mathbb{A}_{\mathbb{Z}}^r$  provides an open immersion  $U \rightarrow \mathbb{A}_{\mathbb{Z}}^r$ . Since  $\mathbb{A}^r(\mathbb{Q})$  is dense in  $\mathbb{A}^r(\overline{\mathbb{Q}})$ , it follows that  $U(\mathbb{Q}) \neq \emptyset$ . As  $U$  is of finite type, any map  $\text{Spec}(\mathbb{Q}) \rightarrow U$  factors as  $\text{Spec}(\mathbb{Q}) \rightarrow \text{Spec}(\mathbb{Z}_f) \rightarrow U$  for some  $f$ . Thus we have a point  $s : \text{Spec}(\mathbb{Z}_f) \rightarrow U$ . We claim that this point satisfies the statement of the lemma. To lighten notation, let us base change to  $\text{Spec}(\mathbb{Z}_f)$ . Consider the centralizer  $H := \text{Cent}(s, G)$ . It is a closed subscheme of  $G$ , hence affine and of finite type over  $\mathbb{Z}_f$ , and contains  $T$ . By generic flatness, we may assume that  $H$  is flat, after possibly changing  $f$ . By the choice of  $s$ , all fibers of  $H$  and  $T$  coincide. By [SGA3, exp. 10, 4.9],  $H$  is a torus, and since  $T$  is a maximal torus, it follows that  $H = T$ .  $\square$

**Corollary 3.2.** *There exists  $f \in \mathbb{Z} \setminus \{0\}$  such that for all schemes  $S \rightarrow \operatorname{Spec}(\mathbb{Z}_f)$  we have*

$$Z(G)(S) = Z(G(S)).$$

*Proof.* The inclusion  $\subset$  is obvious from the definition of  $Z(G)$  and we now have to show the converse. Choose  $f$  and  $s \in T(\mathbb{Z}_f)$  as in the above lemma. Let  $S \rightarrow \operatorname{Spec}(\mathbb{Z}_f)$  and  $x \in Z(G(S))$ . If  $s_S \in T(S)$  denotes the image of  $s$  under  $T(\mathbb{Z}_f) \rightarrow T(S)$ , then

$$x \in \operatorname{Cent}(s_S, G_S)(S) = T(S)$$

We claim that for every root  $\alpha \in R(T_S, G_S)$  we have  $\alpha(x) = 1$ . Assume by way of contradiction that this were not the case. Let  $u_\alpha : \mathbb{G}_{a,S} \rightarrow G_S$  be the root subgroup corresponding to  $\alpha$ , and  $y = u_\alpha(1)$ . Then  $y \in G(S)$  is a point not centralized by  $x$ , contrary to the assumptions. It follows that

$$x \in \bigcap_{\alpha \in R(T_S, G_S)} \ker(\alpha)(S) = Z(G)(S)$$

where the last equality is [SGA3, exp. 22, 4.1.6].  $\square$

**Corollary 3.3.** *There exists a finite set of primes  $P$  such that  $|Z(G_k)|$  divides  $\operatorname{ord}(Z(G))$  for all primes  $p \notin P$ . In particular, if  $m$  is an integer coprime to the elements of  $P$ , then the order of every element of  $Z(G(m))$  divides  $\operatorname{ord}(Z(G))$ .*

*Proof.* The second statement is an immediate consequence of the first, since  $Z(G(m)) = \prod_{p^k \parallel m} Z(G_k)$ . To prove the first, let  $P$  be the set of primes  $p$  for which  $Z(G)(\mathbb{Z}/p^k\mathbb{Z})$  is a proper subgroup of  $Z(G_k)$ . According to Corollary 3.2 the set  $P$  is finite. For a prime  $p$  not in  $P$ , we then have  $Z(G_k) = Z(G)(\mathbb{Z}/p^k\mathbb{Z})$ . As already remarked,  $Z(G)$  is a finite product of  $\mu_n$ 's. Since  $(\mathbb{Z}/p^k\mathbb{Z})^\times$  is cyclic, the number  $|\mu_n(\mathbb{Z}/p^k\mathbb{Z})|$  divides  $n$ . The statements now follow.  $\square$

**Lemma 3.4.** *The natural projection  $\mathbb{Z}/p^k\mathbb{Z} \rightarrow \mathbb{Z}/p^{k-1}\mathbb{Z}$  induces a surjective homomorphism*

$$G_k \rightarrow G_{k-1}$$

*For all but finitely many primes  $p$ , this homomorphism restricts to an isomorphism*

$$Z(G_k) \rightarrow Z(G_{k-1}).$$

*Proof.* The first claim follows directly from the infinitesimal lifting property of smoothness. For the second claim, let  $p$  be a prime which does not divide  $\operatorname{ord}(Z(G))$  and for which  $Z(G_k) = Z(G)(\mathbb{Z}/p^k\mathbb{Z})$  for all  $k$ . By Corollaries 3.2 and 3.3 these are all but finitely many primes. Then  $Z(G)$  is étale over  $\mathbb{Z}_{(p)}$  and this implies the bijectivity of the second map.  $\square$



**Corollary 3.5.** *Assume that  $G$  is simply-connected. Then for all but finitely many  $p$ ,*

$$Z(G_k/Z(G_k)) = \{1\}.$$

*Proof.* We prove this by induction on  $k$ . The base case is  $k = 1$ , which is known, since  $G(\mathbb{F}_p)/Z(G(\mathbb{F}_p))$  is simple. For the induction step, let  $k > 1$ . Let  $z \in G_k$  be an element which is central in  $G_k/Z(G_k)$ . Then for all  $g \in G_k$ ,  $z_g := gzg^{-1}z^{-1} \in Z(G_k)$ . Under the surjection  $G_k \rightarrow G_{k-1}$ , the element  $z$  maps to an element  $\bar{z}$  with the same property. Applying the induction hypothesis we see that  $\bar{z} \in Z(G_{k-1})$ . This implies, that  $\bar{z}_g = 1$ . Lemma 3.4 now implies  $z_g = 1$  and the statement follows.  $\square$

For  $0 \leq i \leq k$ , let  $G_k^i := \ker(G_k \rightarrow G_i)$ . This provides a descending filtration

$$G_k = G_k^0 \geq G_k^1 \geq \dots \geq G_k^k = \{1\}.$$

We fix a closed embedding  $G \rightarrow \mathrm{SL}_m$  defined over  $\mathbb{Z}$ . This yields an embedding of Lie-algebras  $\mathfrak{g} \rightarrow \mathfrak{sl}_m$  defined over  $\mathbb{Z}$ . We identify  $G$  and  $\mathfrak{g}$  with their respective images. Clearly  $G_k^i = [1 + p^i M_m(\mathbb{Z}/p^k \mathbb{Z})] \cap G_k$ , and an element  $1 + p^i x \in G_k^i$  belongs to  $G_k^{i+1}$  if and only if  $x \equiv 0 \pmod{p}$ .

The following Lemma is a well-known result from the theory of Moy-Prasad filtrations [MP94].

**Lemma 3.6** (Moy-Prasad).

1.  $[G_k^i, G_k^j] \subset G_k^{i+j}$ .
2. For  $1 \leq i \leq k-1$  the map

$$G_k^i/G_k^{i+1} \rightarrow \mathfrak{g}(\mathbb{F}_p), \quad 1 + p^i x \mapsto x \pmod{p}$$

*induces an isomorphism of groups, which is equivariant with respect to the action of  $G(\mathbb{F}_p)$  on both sides by conjugation.*

**Remark 3.7.** In particular, one sees inductively that each  $G_k^i$  for  $i > 0$  is a  $p$ -group.

**Lemma 3.8.** *There exists positive constants  $c, C$  such that for all prime powers  $m = p^k$*

$$cm^{\dim(G)} \leq |G(m)| \leq Cm^{\dim(G)}.$$

*Proof.* In the case  $k = 1$  the lemma follows from [S68, Theorem 25, Section 9]. The general case reduces to this, because according to Lemma 3.6 we have  $|G_k| = p^{(k-1)\dim(G)} |G(\mathbb{F}_p)|$ .  $\square$

**Lemma 3.9.** *For all but finitely many  $p$ , the Lie-algebra  $\mathfrak{g}(\mathbb{F}_p)$  has no center, and the adjoint action of  $G(\mathbb{F}_p)/Z(G(\mathbb{F}_p))$  on  $\mathfrak{g}(\mathbb{F}_p)$  is faithful and irreducible.*



*Proof.* This is a well-known classical result. See for example [H84], [H82].  $\square$

**Lemma 3.10.** *Assume that  $p$  is sufficiently large, and  $0 \leq i \leq k-2$ . For every  $g \in G_k^i \setminus G_k^{i+1}Z(G_k)$  there exists  $h \in G_k^1$  such that  $hgh^{-1}g^{-1} \in G_k^{i+1} \setminus G_k^{i+2}Z(G_k)$ .*

*Proof.* Note first that by Lemma 3.4,  $G_k^{i+1} \cap (G_k^{i+2}Z(G_k)) = G_k^{i+2}$ . Hence it is enough to find  $h$  such that  $hgh^{-1}g^{-1} \notin G_k^{i+2}$ .

Write  $h = 1 + py$  with some  $y \in M_m(\mathbb{Z}/p^k\mathbb{Z})$  to be determined. We will make use of the following computation: For any  $x \in M_m(\mathbb{Z}/p^k\mathbb{Z})$  we have

$$\begin{aligned} (1+py)x(1+py)^{-1} &= (x+pyx)(1+py)^{-1} \\ &= (x+pxy-p[x,y])(1+py)^{-1} \\ &= (x-p[x,y])(1+py)^{-1} \end{aligned}$$

where  $[x,y] = xy - yx$ .

First assume that  $i = 0$ . Then using the above computation we see that

$$hgh^{-1}g^{-1} = 1 - p[g,y](1+py)^{-1}g^{-1}$$

Clearly the right hand side belongs to  $G_k^1$ , and to show that it does not belong to  $G_k^2$  it is enough by Lemma 3.6 to show that the reduction mod  $p$  of the matrix  $[g,y](1+py)^{-1}g^{-1} \in M_m(\mathbb{Z}/p^k\mathbb{Z})$  is non-zero. Call this reduction  $T$ . It belongs to  $\mathfrak{g}(\mathbb{F}_p)$ . Using the formula

$$(1+py)^{-1} = \sum_{j=0}^{k-1} (-py)^j$$

we compute that  $T = [\bar{g}, \bar{y}]\bar{g}^{-1} = \bar{g}\bar{y}\bar{g}^{-1} - \bar{y}$ . By Lemma 3.4, the preimage of  $Z(G(\mathbb{F}_p))$  under  $G_k \rightarrow G(\mathbb{F}_p)$  is  $G_k^1Z(G_k)$ . Thus by assumption, the image  $\bar{g}$  of  $g$  in  $G(\mathbb{F}_p)/Z(G(\mathbb{F}_p))$  is non-trivial, and by Lemma 3.9 there exists  $\bar{y} \in \mathfrak{g}(\mathbb{F}_p)$  such that  $\bar{g}\bar{y}\bar{g}^{-1} \neq \bar{y}$ . According to Lemma 3.6, there exists  $h = 1 + py \in G_k^1$  corresponding to this  $\bar{y}$ . This completes the proof in the case  $i = 0$ .

Now assume  $i > 0$ . We write  $g = 1 + p^i x$  for some  $x \in M_m(\mathbb{Z}/p^k\mathbb{Z})$  whose reduction mod  $p$  belongs to  $\mathfrak{g}(\mathbb{F}_p)$ . Then

$$\begin{aligned} &(1+py)(1+p^i x)(1+py)^{-1}(1+p^i x)^{-1} \\ &= (1+p^i(1+py)x(1+py)^{-1})(1+p^i x)^{-1} \\ &= (1+p^i x - p^{i+1}[x,y](1+py)^{-1})(1+p^i x)^{-1} \\ &= 1 - p^{i+1}[x,y](1+py)^{-1}(1+p^i x)^{-1} \end{aligned}$$

Again  $hgh^{-1}g^{-1} \in G_k^{i+1}$ , and we want to choose  $y$  so that this element does not belong to  $G_k^{i+2}$ . By Lemma 3.6, this is equivalent to the demand that the reduction mod  $p$  of the element

$$[x,y](1+py)^{-1}(1+p^i x)^{-1} \in M_m(\mathbb{Z}/p^k\mathbb{Z})$$

be non-trivial. Using the formula for  $(1 + p^i x)^{-1}$  analogous to that used above for  $(1 + py)^{-1}$  we compute that this element is equal mod  $p$  to  $[x, y]$ . Now we consider the image of  $[x, y] \in M_m(\mathbb{F}_p)$ . Of course, this is just the bracket of the images of  $x$  and  $y$  in  $M_m(\mathbb{F}_p)$ . But these images, and hence their bracket, lie in  $\mathfrak{g}(\mathbb{F}_p)$ . Again as in the case  $i = 0$ , specifying  $h$  is equivalent to choosing the class of  $y$  in  $\mathfrak{g}(\mathbb{F}_p)$  in such a way that its bracket with the class of  $x$  is non-trivial. Since the Lie-algebra  $\mathfrak{g}(\mathbb{F}_p)$  has no center, the class of  $x$  is non-central, and so an appropriate  $y$  exists.  $\square$

**Proposition 3.11.** *Assume that  $p$  is sufficiently large and  $G$  is simply-connected. Then every normal subgroup  $N < G_k$  which contains  $Z(G_k)$  equals  $G_k^i Z(G_k)$  for some  $i$ .*

*Proof.* We will first prove under the assumption  $k > 1$  by descending induction on  $i$  the following statement.

$$\forall 0 \leq i < k: \quad N \cap [G_k^i \setminus G_k^{i+1} Z(G_k)] \neq \emptyset \quad \Rightarrow \quad G_k^i \subset N$$

The base case is when  $i = k - 1 > 0$ . Then the isomorphism of Lemma 3.6 identifies  $G_k^i$  with  $\mathfrak{g}(\mathbb{F}_p)$  and  $N \cap G_k^i$  with an invariant subspace of  $\mathfrak{g}(\mathbb{F}_p)$ . By assumption this space is non-trivial, and by Lemma 3.9 it is all of  $\mathfrak{g}(\mathbb{F}_p)$ , hence  $N \cap G_k^i = G_k^i$ . For the induction step, assume  $i \geq 0$ . Let  $g \in N \cap [G_k^i \setminus G_k^{i+1} Z(G_k)]$ . Use Lemma 3.10 to obtain  $h \in G_k^1$  such that  $hgh^{-1}g^{-1} \in G_k^{i+1} \setminus G_k^{i+2} Z(G_k)$ . Then  $hgh^{-1}g^{-1} \in N$ , and we may apply the induction hypothesis to conclude  $G_k^{i+1} \subset N$ . Now look at the normal subgroup  $(N \cap G_k^i)/G_k^{i+1}$  of  $G_k^i/G_k^{i+1}$ . If  $i > 0$ , then we have the isomorphism  $G_k^i/G_k^{i+1} \rightarrow \mathfrak{g}(\mathbb{F}_p)$  and the image of that normal subgroup is a non-trivial invariant subspace. If  $i = 0$ , then we have the isomorphism  $G_k^i/G_k^{i+1} \rightarrow G(\mathbb{F}_p)$  and the image of that normal subgroup is normal subgroup of  $G(\mathbb{F}_p)$  which properly contains  $Z(G(\mathbb{F}_p))$ . In both cases, we conclude that  $(N \cap G_k^i)/G_k^{i+1} = G_k^i/G_k^{i+1}$ , and hence  $N \cap G_k^i = G_k^i$ . This completes the induction.

Now we show how the proposition follows from the above statement. The case  $k = 1$  is trivial since  $G_1/Z(G_1)$  is simple. Thus assume  $k > 1$ . If  $N = Z(G_k)$  there is nothing to prove. Otherwise there exists a unique smallest index  $i$  such that  $G_k^i \setminus G_k^{i+1} Z(G_k)$  contains an element of  $N$ . By the above statement,  $Z(G_k)G_k^i \subset N$ , but by minimality of  $i$  this must in fact be an equality.  $\square$

**Proposition 3.12.** *Let  $N$  be a natural number, and  $H = \ker[G(\mathbb{Z}) \rightarrow G(N)]$ . If  $G$  is simply-connected, then for any  $m$  coprime to  $N$  the projection  $G(\mathbb{Z}) \rightarrow G(m)$  maps  $H$  surjectively onto  $G(m)$ .*

*Proof.* We begin with the special case  $N = 1$ , then  $H = G(\mathbb{Z})$ . Since  $G$  is smooth, the natural projection  $G(\mathbb{Z}_p) \rightarrow G(\mathbb{Z}/p^k\mathbb{Z})$  is surjective for all primes  $p$  and all natural numbers  $k$ , and hence the natural projection  $G(\widehat{\mathbb{Z}}) \rightarrow G(m)$  is surjective for all natural numbers  $m$ . By strong approximation ([PR94]), the inclusion  $G(\mathbb{Z}) \rightarrow G(\widehat{\mathbb{Z}})$  has dense image. Thus, the natural projection  $G(\mathbb{Z}) \rightarrow G(m)$  is surjective.

For the general case, we have  $G(Nm) \cong G(N) \times G(m)$ , and by the first part of the proof, the projection  $G(\mathbb{Z}) \rightarrow G(N) \times G(m)$  is surjective. The preimage in  $G(\mathbb{Z})$  of the subgroup  $1 \times G(m)$  of  $G(N) \times G(m)$  is precisely  $H$ , and maps surjectively onto  $G(m)$ .  $\square$

**Proposition 3.13.** *Assume that the rank of  $G$  is at least 2. Let  $u : \mathbb{G}_a \rightarrow G$  be a root subgroup, and  $X$  a finite generating set for  $G(\mathbb{Z})$ . Then there exists a positive constant  $M$  such that for any positive  $z \in \mathbb{Z}$*

$$\|u(z)\|_X \leq M \log(z).$$

*Proof.* Composing  $u$  with the chosen closed embedding  $G \rightarrow \mathrm{SL}_m$ , and then further with the natural inclusion  $\mathrm{SL}_m \rightarrow \mathrm{M}_m$ , we obtain a morphism of  $\mathbb{Z}$ -schemes

$$u' : \mathbb{A}_{\mathbb{Z}}^1 \rightarrow \mathbb{A}_{\mathbb{Z}}^{m^2}$$

which is given by collection  $\{u'_{i,j}\}$  of  $m^2$ -many polynomials in one variable with integral coefficients. Let  $k = \max \deg(u'_{i,j}) + 1$ . Then there exists a positive constant  $C$  such that  $u'_{i,j}(z) \leq Cz^k$  for all positive integers  $z$  and all  $i, j$ . Thus  $\|u'(z)\| \leq Cz^k$  for all  $z \in \mathbb{N}$ , where  $\|\cdot\|$  is the maximum norm on  $\mathrm{M}_m(\mathbb{R})$ . The result now follows from Theorem A in [LMR01].  $\square$

We are now ready to prove our main lower bound. In the proof, we are going to use the fact that if  $G$  is simply-connected and has rank at least 2, then  $G(\mathbb{Z})$  has the congruence subgroup property. We refer the reader to [PR94, Chap. 9.5] for a discussion of this property. Also recall that a subgroup of  $G(m)$  is called *essential* if it does not contain the kernel of the natural map  $G(m) \rightarrow G(r)$  for any  $r|m$  with  $r < m$ .

**Theorem 3.14.** *Assume that the rank of  $G$  is at least 2. Let  $K$  be a number field,  $f \in \mathcal{O}_K$ , and  $\Delta$  a finitely generated subgroup of  $G(\mathbb{C})$  with the property that  $\Delta \cap G(\mathcal{O}_{K,f})$  is of finite-index in  $G(\mathcal{O}_{K,f})$ . Then*

$$F_{\Delta}(n) \succeq n^{\dim(G)}.$$

*Proof.* Let  $G_{\mathrm{sc}}$  be the simply connected cover of  $G$ , and  $p : G_{\mathrm{sc}}(\mathcal{O}_{K,f}) \rightarrow G(\mathcal{O}_{K,f})$  the natural map. Then  $\Delta_{\mathrm{sc}} := p^{-1}(\Delta \cap G(\mathcal{O}_{K,f}))$  is of finite index in  $G_{\mathrm{sc}}(\mathcal{O}_{K,f})$  and the map  $p : \Delta_{\mathrm{sc}} \rightarrow \Delta$  has finite kernel. By Lemma 2.4 we may assume for the rest of the proof that  $G = G_{\mathrm{sc}}$  and  $\Delta \subset G(\mathcal{O}_{K,f})$ .

Since  $\Delta$  is of finite-index in  $G(\mathcal{O}_{K,f})$ , so is  $\Delta \cap G(\mathbb{Z})$  of finite-index in  $G(\mathbb{Z})$ . By virtue of the congruence subgroup property of  $G(\mathbb{Z})$ , we can find a principal congruence subgroup  $\Delta' \subset \Delta \cap G(\mathbb{Z})$ . Applying again Lemma 2.4, we may assume for the rest of the proof that  $\mathcal{O}_{K,f} = \mathbb{Z}$  and that  $\Delta$  is a principal congruence subgroup of  $G(\mathbb{Z})$ .

Let  $N = \mathrm{ord}(Z(G))$ . By Lemma 2.1, it suffices to find a lower bound for  $F_{\Delta}^N$ . Loosely speaking, we will see that working with  $F_{\Delta}^N$  instead of  $F_{\Delta}$  will aid us in ignoring certain central elements in finite images of  $\Delta$ .

We first construct candidates that are poorly approximated by finite quotients. Let  $X$  and  $Y$  be finite generating sets for  $G(\mathbb{Z})$  and  $\Delta$  respectively. Let  $S$  be the set of primes  $p$  for which at least one of the following conditions fails

- $|Z(G_k)|$  divides  $N$ ,
- If  $Z(G_k) \trianglelefteq N \trianglelefteq G_k$  then  $N = G_k^i Z(G_k)$  for some  $i$ ,
- The projection  $G(\mathbb{Z}) \rightarrow G_k$  maps  $\Delta$  surjectively onto  $G_k$ .

where as before  $G_k = G(\mathbb{Z}/p^k\mathbb{Z})$ . By Corollary 3.3 and Propositions 3.11 and 3.12 this set is finite. Put  $\alpha = \prod_{p \in S} p$  and  $r_k = \alpha^k \text{lcm}(1, \dots, k)$ . Let  $u : \mathbb{G}_a \rightarrow G$  be a root subgroup, and  $B_k = u(r_k)$ . Since  $u$  is defined over  $\mathbb{Z}$ , we have  $B_k \in G(\mathbb{Z})$ , hence  $A_k := B_k^{[G(\mathbb{Z}) : \Delta]} \in \Delta$ . The elements  $A_k$  will be our candidates for achieving lower bounds for  $F_\Delta^N$ .

Next, we bound the word length of  $A_k$ , i.e. the function  $k \mapsto \|A_k\|_Y$ . By Proposition 3.13 there exists a natural number  $M$  such that

$$\|A_k\|_X \leq M \log(\text{lcm}(1, \dots, k) \alpha^k).$$

Hence, by the prime number theorem we may find a potentially different natural number  $M$  so that  $\|A_k\|_X \leq Mk$ . Finally, since  $G(\mathbb{Z})$  is quasi-isometric to  $\Delta$ , we have that

$$\|A_k\|_Y \leq Mk, \tag{1}$$

for a some other natural number  $M$ .

The remainder of the proof is devoted to finding a lower bound for the cardinality of any finite quotient  $Q = \Delta/H$  which detects  $A_k^N$ , in particular to the quotient realizing  $D_\Delta^N(A_k)$ . We start by taking one such quotient  $Q$ . Since we are looking for a lower bound of the cardinality of  $Q$ , we may replace it by either a subgroup or a quotient of it, and we will do so repeatedly in the following.

By the congruence subgroup property for  $G(\mathbb{Z})$  there exists a natural number  $m$  such that the kernel of the projection  $\phi : G(\mathbb{Z}) \rightarrow G(m)$  lies in  $H$ . Let  $\Delta', H'$ , and  $A'_k$  be the images of  $\Delta$ ,  $H$ , and  $A_k$  respectively in  $G(m)$ . By the Chinese remainder theorem, we may write  $G(m) = A \times B$  where

$$A = \prod_{\substack{p^j \parallel m \\ p \in S}} G(p^j) \quad \text{and} \quad B = \prod_{\substack{p^j \parallel m \\ p \notin S}} G(p^j).$$

and  $p^j \parallel m$  means that  $j$  is the greatest power of  $p$  which divides  $m$ .

We know  $(A'_k)^N \neq 1$ . For any  $c \in Z(B)$  we have  $\text{ord}(c) \mid N$  (see Corollary 3.3 and the choices of  $S$  and  $N$ ). Thus we have  $(cA'_k)^N = (A'_k)^N$  for any  $c \in Z(B)$ , which implies  $cA'_k \notin$

$H'$ . Hence,  $A'_k \notin H'Z(B)$ . Letting  $A''_k$ ,  $\Delta''$ , and  $H''$  be the images of  $A'_k$ ,  $\Delta'$ , and  $H'$  in  $A \times B/Z(B)$  respectively, we have that  $A''_k \notin H''$ . Further,  $[\Delta'' : H''] \leq [\Delta' : H']$  since  $\Delta''/H''$  is an image of  $Q = \Delta'/H'$ .

We claim that any quotient of  $B/Z(B)$  is centerless: Indeed, by the choice of  $S$ , for every  $p \notin S$ , Lemma 3.4 and Corollary 3.5 imply that all quotients of  $G(p^j)/Z(G(p^j))$  are centerless. By [LL01, 1.4] every normal subgroup of  $B/Z(B)$  is a product of normal subgroups of the factors of  $B/Z(B)$ , and the statement follows.

Recall that  $\Delta$  was assumed to be a principal congruence subgroup of  $G(\mathbb{Z})$ . By Proposition 3.12,  $G(\mathbb{Z})$  projects onto  $A \times B/Z(B)$ . Hence,  $\Delta''$  is normal in  $A \times B/Z(B)$ , and applying [LL01, 1.3, 1.4] we see that  $\Delta'' = \Delta_1 \times \Delta_2$ , where

$$\Delta_1 = \pi_1(\Delta'') \quad \text{and} \quad \Delta_2 = \pi_2(\Delta''),$$

where  $\pi_1$  and  $\pi_2$  are the natural projection maps of  $A \times B/Z(B)$  onto  $A$  and  $B/Z(B)$  respectively.

By the choice of  $S$  we have  $\Delta_2 = B/Z(B)$ . The subgroup  $H''$  is normal in  $\Delta'' = \Delta_1 \times \Delta_2$ , and since  $\Delta_2$  has no center, [LL01, Corollary 1.4] applies again giving  $H'' = H_1 \times H_2$  where  $H_1 = \pi_1(H'')$  and  $H_2 = \pi_2(H'')$ . Now since  $A''_k \notin H_1 \times H_2$  we have two cases:  $\pi_1(A''_k) \notin \pi_1(H'')$  or  $\pi_2(A''_k) \notin \pi_2(H'')$ . In both cases, we claim that there exists a natural number  $M$ , independent of  $k$ , such that  $M|Q| \geq k^d$ , where  $d := \dim(G)$ .

We first handle the case  $\pi_1(A''_k) \notin \pi_1(H'')$ . Write  $A = G(m_0)$ , let  $r$  be the smallest natural number such that the kernel of the natural map  $\phi : G(m_0) \rightarrow G(r)$  is contained in  $\pi_1(H'')$ . Then  $\phi(\pi_1(A''_k)) \notin \phi(\pi_1(H''))$  and  $\phi(\pi_1(H''))$  is essential or trivial. Since the image of  $A_k$  in  $G(r)$  is nontrivial,  $r$  does not divide  $\alpha^k$ . But any prime dividing  $r$  also divides  $\alpha$  (recall the choices of  $A$ ,  $r$  and  $\alpha$ ), hence  $p^k | r$  for some  $p \in S$ . In the case  $\phi(\pi_1(H''))$  is essential, [LS03, Proposition 6.1.2] gives  $C[G(r) : \phi(\pi_1(H''))] \geq r \geq p^k$ , where  $C$  is a natural number that only depends on  $G$ . If  $\phi(\pi_1(H''))$  is trivial, we get the better bound  $C|G(r)| \geq C|G(p^k)| \geq p^{kd}$  by Lemma 3.8 where  $C$  is again a natural number that depends only on  $G$ . Set  $M' = C[G(\mathbb{Z}) : \Delta]$ . Since  $[G(r) : \phi(\pi_1(\Delta''))] \leq [G(\mathbb{Z}) : \Delta]$ , we have

$$M'[\Delta'' : H''] \geq C[G(r) : \phi(\pi_1(\Delta''))][\phi(\pi_1(\Delta'')) : \phi(\pi_1(H''))] = C[G(r) : \phi(\pi_1(H''))] \geq p^k.$$

There exists a natural number  $M''$  such that  $M''p^k \geq k^d$  for all  $p \in S$  and  $k \in \mathbb{N}$ . Setting  $M = M'M''$ , we see that

$$M[\Delta'' : H''] \geq M''p^k \geq k^d.$$

Since  $|Q| \geq [\Delta'' : H'']$ , the claim is shown.

Next we handle the case  $\pi_2(A''_k) \notin \pi_2(H'')$ . By repeated use of Corollary 1.4 in [LL01], there exists a natural projection  $\phi : A \times B/Z(B) \rightarrow G_k/Z(G_k)$  with  $\phi(A''_k) \notin \phi(H'')$  and  $G_k = G(p^k)$  where  $p \notin S$ . By Proposition 3.11 and the normality of  $H_2$  in  $\Delta_2 = B/Z(B)$ , we have  $\phi(H'') = G_k^i/Z(G_k)$  for some  $i$ , hence the image of  $\phi(A''_k)$  through the natural projection onto  $G_i/Z(G_i)$  is non-trivial. Further,  $Q$  maps onto  $G_i/Z(G_i)$ .

From the estimate  $M'|G_i| \geq p^{di}$  (Lemma 3.8), where  $M'$  is a natural number, and the fact that  $p^i$  does not divide  $\text{lcm}(1, \dots, k)$  we obtain  $p^i \geq k$ , and thus,

$$M'|G_i| \geq p^{id} \geq k^d.$$

Finally, since  $|G_i|/|Z(G_i)| \leq |Q|$  and  $|Z(G_i)| \leq N$  (by the choice of  $S$ ), the claim holds with  $M = M'N$ .

The inequality  $M|Q| \geq k^d$  in tandem with Inequality (1) gives some natural number  $M$  such that  $MF_{\Delta}^N(k) \geq k^d$ , finishing the proof of the theorem.  $\square$

## 4 Upper bounds

In this section,  $G$  continues to be a Chevalley group. Our main upper bound result is a corollary of the following three propositions.

**Proposition 4.1.** *Let  $L$  be a number field with ring of integers  $\mathcal{O}_L$ . Then*

$$F_{\mathcal{O}_L}(n) \approx \log(n).$$

*Moreover, the finite quotients of the form  $\mathbb{Z}/p\mathbb{Z} \cong \mathcal{O}_L/\mathfrak{p}$ , where  $p$  is a prime number that splits completely in  $\mathcal{O}_L$ ,  $\mathfrak{p}|p\mathcal{O}_L$ , are enough to obtain the upper bound.*

*Proof.* The fact  $F_{\mathcal{O}_L}(n) \succeq \log(n)$  follows immediately from [B10, Thm. 2.2] and Lemma 2.4. Thus it is enough to prove  $F_{\mathcal{O}_L}(n) \preceq \log(n)$ .

Let  $S = \{b_1, \dots, b_k\}$  be an integral basis for  $\mathcal{O}_L$ , and fix a nontrivial  $g$  in  $\mathcal{O}_L$  with  $\|g\|_S = n$ . Then  $g = \sum_{i=1}^n a_i b_i$  where  $a_i \in \mathbb{Z}$  and  $|a_i| \leq n$ . Since  $g \neq 0$  there exists  $k$  such that  $a_k \neq 0$ . By the Cebotarëv density theorem, the set  $P$  of all primes in  $\mathbb{Z}$  that split in  $\mathcal{O}_L$  has nonzero natural density in the set of all primes. We claim that there exists  $C > 0$ , which does not depend on  $n$ , and a prime  $q$  such that  $(q)$  splits in  $\mathcal{O}_L$  and  $q \leq C \log(n)$  and  $a_k \not\equiv 0 \pmod{q}$ . Indeed, enumerate  $P = \{q_1, q_2, \dots\}$ . Let  $q_{r+1}$  be the first prime in  $P$  such that  $a_k \not\equiv 0 \pmod{q_{r+1}}$ . Then  $q_1 \cdots q_r$  divides  $a_k$  and by the prime number theorem and positive density of  $P$ , we have that  $q_{r+1} \leq Mr \log(r)$  for some  $M > 0$ , depending only on  $L$ . A similar calculation shows that there exists  $M' > 0$  such that  $q_1 \cdots q_r \geq e^{M' r \log(r)}$ . Hence,  $q_{r+1} \leq C \log(a_k)$ , where  $C > 0$  depends only on  $L$ . The claim is shown.

Write  $(q) = \mathfrak{q}_1 \cdots \mathfrak{q}_c$  with  $|\mathcal{O}_L/\mathfrak{q}_i| = q$ . Since  $q$  does not divide  $a_k$  and since the integral basis  $S$  gets sent to a  $\mathbb{F}_q$ -basis of  $\mathcal{O}_L/(q)$ , we have that  $g \neq 1$  in  $\mathcal{O}_L/(q)$ . Hence, there exists one  $\mathfrak{q}_i$  with  $g \neq 1$  in  $\mathcal{O}_L/\mathfrak{q}_i$ . As the cardinality of  $\mathcal{O}_L/\mathfrak{q}_i$  is equal to  $q$  which is no greater than  $C \log(n)$ , we have the desired upper bound.  $\square$

**Proposition 4.2.** *Let  $\Gamma$  be a finitely generated subgroup of  $G(\mathcal{O}_{L,f})$ , where  $L$  is a number field and  $f \in \mathbb{Z}$ . Then*

$$F_{\Gamma} \preceq n^{\dim(G)}.$$

*Proof.* Recall that we have fixed a closed embedding  $G \rightarrow \mathrm{SL}_m$  and are identifying  $G$  with its image. Let  $\mathcal{X}$  be a finite set of generators for  $\Gamma$  as a semigroup. Let  $S$  be an integral basis for  $\mathcal{O}_L$ . We claim that there exists  $\lambda > 0$  such that for any  $A \in \Gamma$  with  $\|A\|_{\mathcal{X}} = n$  and any non-zero coefficient  $a \in \mathcal{O}_{L,f}$  of  $A - I$  we have

$$\|f^k a\|_S \leq \lambda^n$$

where  $k$  is the least natural number such that  $f^k a \in \mathcal{O}_L$ .

To prove the claim, let  $a' = a + 1$  or  $a' = a$  according to whether  $a$  is a diagonal coefficient or not. Thus  $a'$  is a coefficient of  $A$ . Let  $K$  be the least natural number such that for all  $X \in \mathcal{X}$ ,  $f^K X \in M_m(\mathcal{O}_L)$ . Because  $A$  is a product of exactly  $n$  elements of  $X$ , we have  $f^{nK} A \in M_m(\mathcal{O}_L)$ , and in particular  $k < nK$ . Then

$$\|f^k a\|_S \leq \|f^{nK} a\|_S \leq \|f^{nK} a'\|_S + f^{nK} \|1\|_S.$$

This reduces the above claim to the following. There exists  $\mu > 0$  such that for any  $A \in \Gamma$  with  $\|A\|_{\mathcal{X}} = n$  and any non-zero coefficient  $a \in \mathcal{O}_{L,f}$  of  $A$  we have

$$\|f^{nK} a\|_S \leq \mu^n.$$

We claim that if  $\alpha$  denotes the maximum of  $\|st\|_S$ , where  $s, t$  range over the elements of  $S$ , and  $\beta$  denotes the maximum of  $\|f^K x\|_S$ , where  $x$  ranges over all entries of all elements of  $\mathcal{X}$ , then  $\mu := m\alpha\beta$  satisfies the last statement. To see this, consider first the case  $A = XY$  with  $X, Y \in \mathcal{X}$ . The entries of  $A$  are scalar products of the rows of  $X$  and the columns of  $Y$ . Thus we are led to study  $\|x \cdot y\|_S$  for  $x, y \in \mathcal{O}_L^m$ , where  $\cdot$  denotes scalar product. Clearly we have  $\|x \cdot y\|_S \leq m \max\{\|x_i y_i\|_S : 1 \leq i \leq m\}$ . In terms of the basis  $S$  we can write

$$x_i = \sum_{s \in S} \lambda_{x,i,s} s \quad \text{and} \quad y_i = \sum_{s \in S} \lambda_{y,i,s} s$$

where the  $\lambda$ 's belong to  $\mathbb{Z}$ . One computes

$$\|x_i y_i\|_S \leq \|x_i\|_S \|y_i\|_S \max\{\|st\|_S : s, t \in S\}.$$

This formula and induction on  $n$  complete the proof of the claim.

To complete the proof of the proposition, let  $A \in \Gamma$  be such that  $\|A\|_{\mathcal{X}} \leq n$ . Let  $a$  be a non-zero entry of  $A - I$  and  $k$  the least integer with  $f^k a \in \mathcal{O}_L$ . According to Proposition 4.1 and the claim above there exists a natural number  $M$ , independent of  $n$ , and a homomorphism  $\phi : \mathcal{O}_L \rightarrow \mathbb{F}_p$  such that  $p < Mn$  and  $\phi(f^k a) \neq 0$ . For all but finitely many primes  $p$ , we have that  $\phi(f)$  is non-zero in  $\mathbb{F}_p$ . Hence, we may assume that  $\phi$  extends to a homomorphism  $\phi : \mathcal{O}_{L,f} \rightarrow \mathbb{F}_p$  and  $\phi(a) \neq 0$ . The image of  $A$  under the induced map  $G(\mathcal{O}_{L,f}) \rightarrow G(\mathbb{F}_p)$  is non-trivial. Further, according to Lemma 3.8, there exists  $M' > 0$  such that  $|G(\mathbb{F}_p)| \leq M' p^{\dim(G)}$ . Hence,  $|G(\mathbb{F}_p)| \leq M'(Mn)^{\dim(G)}$ .  $\square$



**Proposition 4.3.** *Let  $K \subset \mathbb{C}$  be a number field,  $b \in \mathcal{O}_K \setminus \{0\}$ , and  $\Gamma \subset G(\mathbb{C})$  a finitely generated subgroup, such that  $G(\mathcal{O}_{K,b}) \cap \Gamma$  is of finite-index in  $\Gamma$ . Then there exists a finite extension  $L \subset \mathbb{C}$  of  $K$ , an element  $f \in \mathbb{Z} \setminus \{0\}$ , and a homomorphism  $\Gamma \rightarrow G(\mathcal{O}_{L,f})$  with finite kernel.*

*Proof.* Let  $S \subset \Gamma$  be a finite generating set. There exists a field  $F \subset \mathbb{C}$ , finitely generated over  $K$ , such that  $S \subset G(F)$ . Let  $t_1, \dots, t_n$  be a transcendence basis for  $F/K$ . The extension  $F/K(t_1, \dots, t_n)$  is finitely generated and algebraic, hence finite. Let  $a \in F$  be a primitive element for that extension. Thus  $F = K(t_1, \dots, t_n, a)$ . The ring  $\mathcal{O}_{K,b}[t_1, \dots, t_n]$  is a free polynomial algebra over  $\mathcal{O}_{K,b}$  with field of fractions  $K(t_1, \dots, t_n)$ . There exists  $s \in \mathcal{O}_K[t_1, \dots, t_n]$  such that the coefficients of the minimal polynomial of  $a$  over  $K(t_1, \dots, t_n)$  lie in the localization  $\mathcal{O}_{K,b}[t_1, \dots, t_n]_s$ . Thus the element  $a$  is integral over  $\mathcal{O}_{K,b}[t_1, \dots, t_n]_s$  and the ring  $\mathcal{O}_{K,b}[t_1, \dots, t_n]_s[a] \subset F$  has  $F$  as its field of fractions. Thus there exists  $r \in \mathcal{O}_{K,b}[t_1, \dots, t_n]_s[a]$ , such that if we put  $R = \mathcal{O}_{K,b}[t_1, \dots, t_n]_s[a]_r$ , then  $S \subset G(R)$ , and consequently  $\Gamma \subset G(R)$ .

We can find a homomorphism of  $\mathcal{O}_{K,b}$ -algebras

$$\phi : \mathcal{O}_{K,b}[t_1, \dots, t_n] \rightarrow \mathcal{O}_{K,b}$$

such that  $\phi(s) \neq 0$ . Then  $\phi$  extends to a homomorphism

$$\phi : \mathcal{O}_{K,b}[t_1, \dots, t_n]_s \rightarrow \mathcal{O}_{K,b\phi(s)}.$$

There exists a finite extension  $L \subset \mathbb{C}$  of  $K$  such that the composition of  $\phi$  with the natural inclusion  $\mathcal{O}_{K,b\phi(s)} \rightarrow K$  extends to a homomorphism

$$\phi : \mathcal{O}_{K,b}[t_1, \dots, t_n]_s[a] \rightarrow L.$$

The element  $\phi(a) \in L$  is integral over  $\mathcal{O}_{K,b\phi(s)}$ , and hence belongs to  $\mathcal{O}_{L,b\phi(s)}$ . Thus in fact we obtain a homomorphism

$$\phi : \mathcal{O}_{K,b}[t_1, \dots, t_n]_s[a] \rightarrow \mathcal{O}_{L,b\phi(s)}.$$

We consider  $\phi(r) \in \mathcal{O}_{L,b\phi(s)}$ . Perturbing  $\phi$  slightly if necessary, we may assume that  $\phi(r) \neq 0$ . In this way we obtain a homomorphism of  $\mathcal{O}_K$ -algebras

$$\phi : R \rightarrow \mathcal{O}_{L,b\phi(rs)}.$$

The algebra homomorphism  $\mathcal{O}_L \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow L$  given by multiplication is an isomorphism. Since  $\mathbb{Q} = \varinjlim_{f \in \mathbb{Z}} \mathbb{Z}_f$ , we conclude that

$$L \cong \varinjlim_{f \in \mathbb{Z}} \mathcal{O}_L \otimes_{\mathbb{Z}} \mathbb{Z}_f \cong \varinjlim_{f \in \mathbb{Z}} \mathcal{O}_{L,f}.$$

Thus there exists some  $f \in \mathbb{Z}$  such that  $[b\phi(rs)]^{-1} \in \mathcal{O}_{L,f}$ . Composing  $\phi$  with the inclusion  $\mathcal{O}_{L,b\phi(rs)} \rightarrow \mathcal{O}_{L,f}$  we finally arrive at a homomorphism of  $\mathcal{O}_{K,b}$ -algebras

$$\phi : R \rightarrow \mathcal{O}_{L,f}.$$

It induces a group homomorphism  $\phi_* : G(R) \rightarrow G(\mathcal{O}_{L,f})$  which fits into the commutative diagram

$$\begin{array}{ccc} G(R) & \xrightarrow{\phi_*} & G(\mathcal{O}_{L,f}) \\ & \nwarrow \quad \nearrow & \\ & G(\mathcal{O}_{K,b}) & \end{array}$$

The restriction of  $\phi_*$  to  $\Gamma$  is the desired homomorphism: Its kernel has trivial intersection with  $G(\mathcal{O}_{K,b})$ , i.e. it avoids a finite-index subgroup of  $\Gamma$ , and hence must be finite.  $\square$

**Corollary 4.4.** *Let  $\Gamma \subset G(\mathbb{C})$  be a finitely generated subgroup. Assume that there exists a finite extension  $K \subset \mathbb{C}$  of  $\mathbb{Q}$  and  $b \in \mathcal{O}_K \setminus \{0\}$  such that  $G(\mathcal{O}_{K,b}) \cap \Gamma$  is of finite-index in  $\Gamma$ . Then*

$$F_\Gamma(n) \preceq n^{\dim(G)}.$$

*Proof.* This follows immediately from Proposition 4.3, Lemma 2.4 and Proposition 4.2.  $\square$

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